Determinants

1. Show that:

(a) $\begin{vmatrix} 2 & 2 & 2 \\ 3 & 1 & 1 \\ 4 & 2 & 1 \end{vmatrix} = 4$ **(b)** $\begin{vmatrix} 4 & 1 & 5 \\ 8 & 2 & 6 \\ 12 & 3 & 7 \end{vmatrix} = 0$

(c) $\begin{vmatrix} 0 & b & c \\ b & 0 & a \\ c & a & 0 \end{vmatrix} = 2abc$ **(d)** $\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = -(a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab)$

(e) $\begin{vmatrix} 4 & 3 & 5 \\ 1 & 6 & 1 \\ 7 & -2 & 8 \end{vmatrix} = -23$ **(f)** $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$

2. Prove the follow identities:

(a) $\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$ **(b)** $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b - c)(c - a)(a - b)$

(c) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a + b + c)(a - b)(b - c)(c - a)$ **(d)** $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = (xy + yz + zx)(x - y)(y - z)(z - x)$

(e) $\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(b + c)(c + a)(a + b)$

3. If ω is one of the imaginary cube roots of unity, find the value of:

(a) $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$ **(b)** $\begin{vmatrix} 1 & \omega^3 & \omega^2 \\ \omega^3 & 1 & \omega \\ \omega^2 & \omega & 1 \end{vmatrix}$

4. Express $\begin{vmatrix} a^3 & b^3 & c^3 \\ a & b & c \\ b+c-a & a+c-b & a+b-c \end{vmatrix}$ as a product of linear factors.

5. Show that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a + b + c)^3$

(Hint : Add the second and third rows to the first row.)

6. Factorize : $\begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix}$.

7. Prove that: $\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix} = (a - b)(b - c)(c - a)(a + b + c)(a^2 + b^2 + c^2)$.

8. By multiplying the first row by abc, and taking factors out of the resulting columns, prove that:

$$\begin{vmatrix} 1 & 1 & 1 \\ bc(c-b) & ca(a-c) & ab(b-a) \\ b^2c & c^2a & a^2b \end{vmatrix} = abc\,(a^3 + b^3 + c^3 - 3abc).$$

9. Let $D(x) = \begin{vmatrix} r_1+x & a+x & a+x \\ b+x & r_2+x & a+x \\ b+x & b+x & r_3+x \end{vmatrix}$ and $f(x) = (r_1 - x)(r_2 - x)(r_3 - x)$.

Express $D(x)$ in the form $D(0) + x\,\Delta$, where Δ is a determinant independent of x.

Hence or otherwise, show that: $D(0) = \dfrac{af(b) - bf(a)}{a-b}$.

10. Explain the fallacy in the following argument:

Let $D(a, b, c) = \begin{vmatrix} a(b+c) & b(c+a) & c(a+b) \\ bc & ca & ab \\ 1 & 1 & 1 \end{vmatrix}$

Since $D(0, b, c) = 0$, a is a factor of D and similarly b and c are factors.
Since $D(a, b, b) = 0$, $(b - c)$ is a factor of D and similarly $(c - a)$ and $(a - b)$ are factors.
But $abc\,(b - c)(c - a)(a - b)$ us a homogeneous polynomial of degree 6, whereas the expansion of D is clearly a homogeneous polynomial of degree 4.

11. Show that $\begin{vmatrix} \sin x & \sin y & \sin z \\ \sin 2x & \sin 2y & \sin 2z \\ \sin 3x & \sin 3y & \sin 3z \end{vmatrix} = 8 \sin x \sin y \sin z (\cos y - \cos z)(\cos z - \cos x)(\cos x - \cos y)$.

12. Prove that $\begin{vmatrix} \cos(\theta+x) & \sin(\theta+x) & 1 \\ \cos(\theta+y) & \sin(\theta+y) & 1 \\ \cos(\theta+z) & \sin(\theta+z) & 1 \end{vmatrix}$ is independent of θ.

13. If $z = (1 + a^2)\,x$, where z is given by $\begin{vmatrix} z & 2a & 1-a^2 \\ 1-a^2 & z & 2a \\ 1-a^2 & 2a & z \end{vmatrix} = 0$, and $a = \dfrac{1}{\sqrt{3}}$,

show that two possible values of x are $\sin\dfrac{\pi}{6}$ and $\sin\dfrac{\pi}{3}$, and find the third value.

14. Show that $\begin{vmatrix} 1 & a & a^2 \\ \cos(n-1)x & \cos nx & \cos(n+1)x \\ \sin(n-1)x & \sin nx & \sin(n+1)x \end{vmatrix} = (1 - 2a\cos x + a^2)\sin x$.

15. Show that $\begin{vmatrix} 1 & \cos A - \sin A & \cos A + \sin A \\ 1 & \cos B - \sin B & \cos B + \sin B \\ 1 & \cos C - \sin C & \cos C + \sin C \end{vmatrix} = 2\begin{vmatrix} 1 & \cos A & \sin A \\ 1 & \cos B & \sin B \\ 1 & \cos C & \sin C \end{vmatrix}$

(Hint: The determinant on the left hand side may be written as the sum of four determinants.)

16. If $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ \sin A & \sin B & \sin C \\ \cos A & \cos B & \cos C \end{vmatrix}$, prove, by expanding from the first row, that:

Δ = sin (A – B) + sin (B – C) + sin (C – A).

By subtracting columns before expansion and by converting differences into products, show that:

$$\Delta = -4\sin\frac{1}{2}(A-B)\sin\frac{1}{2}(B-C)\sin\frac{1}{2}(C-A).$$

17. Prove that $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = \begin{vmatrix} a_1x_1+b_1y_1 & a_2x_1+b_2y_1 \\ a_1x_2+b_1y_2 & a_2x_2+b_2y_2 \end{vmatrix}$.

18. By squaring the determinant: $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$ prove that:

$$\begin{vmatrix} 3 & a+b+c & a^2+b^2+c^2 \\ a+b+c & a^2+b^2+c^2 & a^3+b^3+c^3 \\ a^2+b^2+c^2 & a^3+b^3+c^3 & a^4+b^4+c^4 \end{vmatrix} = (a-b)^2(b-c)^2(c-a)^2.$$

19. Verify that $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = -\begin{vmatrix} a & c & b \\ c & b & a \\ b & a & c \end{vmatrix}$, and deduce that $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}^2 = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$.

20. Given that $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$ is factor of $\begin{vmatrix} 2ab & ac+b^2 & bc+a^2 \\ ab+b^2 & 2bc & ab+c^2 \\ bc+a^2 & ab+c^2 & 2ac \end{vmatrix}$, find the other factor as a determinant of the third order.

21. If $s_n = \alpha^n + \beta^n + \gamma^n$, express $\begin{vmatrix} s_0 & s_2 & s_3 \\ s_4 & s_6 & s_7 \\ s_6 & s_8 & s_9 \end{vmatrix}$ as a product of two determinants of the third order.

22. By multiplying the determinants $\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$, $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 0 & 0 & 0 \end{vmatrix}$, prove that $\begin{vmatrix} 2a^2 & b^2+ab & c^2+ca \\ a^2+ab & 2b^2 & c^2+bc \\ a^2+ca & b^2+bc & 2c^2 \end{vmatrix} = 0$

23. Given a matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ a+b & b+c & c+a \\ ab & bc & ca \end{bmatrix}$, show that the determinant det A = – (b – c)(c – a)(a – b).

Find the matrix X such that XA = B , where

$$B = \begin{bmatrix} x^2+(a+b)x+ab & x^2+(b+c)x+bc & x^2+(c+a)x+ca \\ y^2+(a+b)y+ab & y^2+(b+c)y+bc & y^2+(c+a)y+ca \\ z^2+(a+b)z+ab & z^2+(b+c)z+bc & z^2+(c+a)z+ca \end{bmatrix}.$$

Hence, or otherwise, factorize the determinant det B .

24. If A, H, … , C denote the cofactors a, h, … , c in the determinant $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$, show that:

(a) $\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \Delta^2$, (b) $BC - F^2 = a\,\Delta$.

(Hint : In (a) consider $\Delta \times \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$, in (b) consider $\Delta \times \begin{vmatrix} 1 & H & G \\ 0 & B & F \\ 0 & F & C \end{vmatrix}$.)

25. If the cofactors of elements of the determinant $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ are denoted by the corresponding capital letters, prove that $GH - AF = f\,\Delta$.

26. Solve for x in the equation : $\begin{vmatrix} 1 & a+bcx & bc/a \\ 1 & b+cax & ca/b \\ 1 & c+abx & ab/c \end{vmatrix} = 0.$, $a, b, c \neq 0$

27. Find the general value of θ which satisfies the equation $\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 2\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 2\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 2\theta \end{vmatrix} = 0$.

28. If a, b, c are distinct non-zero numbers, solve the equation $\begin{vmatrix} x & x^2 & a^3-x^3 \\ b & b^2 & a^3-b^3 \\ c & c^2 & a^3-c^3 \end{vmatrix} = 0.$

29. Factorize $\begin{vmatrix} a & b & b \\ b & a & b \\ b & b & a \end{vmatrix}$. If $a = 3x + 1$, $b = 2x + 3$, solve the equation $\Delta = 0$.

30. Prove that, if $x = a^2 + 2bc$, $y = b^2 + 2ca$, $z = c^2 + 2ab$, $s = a^2 + b^2 + c^2$, $p = bc + ca + ab$,

then $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}^2 = \begin{vmatrix} x & z & y \\ y & x & z \\ z & y & x \end{vmatrix} = \begin{vmatrix} s & p & p \\ p & s & p \\ p & p & s \end{vmatrix}$.

31. Express the area of ΔABC as a determinant, where A, B, C are points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$.
If A, B, C are points $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ respectively and OA, OB, OC are adjacent edges of a parallelepiped, express the volume of the parallelepiped as a determinant.

32. Prove that the equation of the line passing through the points $(x_1, y_1), (x_2, y_2)$ is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$

33. Prove that $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = 0$ is a necessary condition for the three lines

$A_ix + B_iy + C_i = 0$, $i = 1, 2, 3$, to be concurrent. Is the condition sufficient ?